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Report Title: Theoretical Analysis of Wave Propagation in Lumped Systems Incorporating a Negative-Stiffness Element

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ARO Progress Report: Theoretical Analysis of Wave Propagation in Lumped Systems Incorporating a Negative-Stiffness Element

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August 14, 2015

This report will summarize progress to date in the theoretical analysis of the effect of a negative-stiffness element on wave propagation. We are initially analyzing lumped systems to minimize the mathematical complexity and thus make the concepts and physical phenomena as clear as possible.

Before showing our progress on incorporating a negative-stiffness element, it is useful to summarize briefly the response of simpler lumped systems to wave propagation, first without damping and then including it.

1 Purely Elastic Spring-Mass Chain

We first treat an infinite spring-mass chain, as analyzed in e.g. Brillouin [1] and Born and Huang [2]. The chain is comprised of equal masses of mass m , equal springs of spring constant k , and unit spacing between the masses, as illustrated in Figure 1.

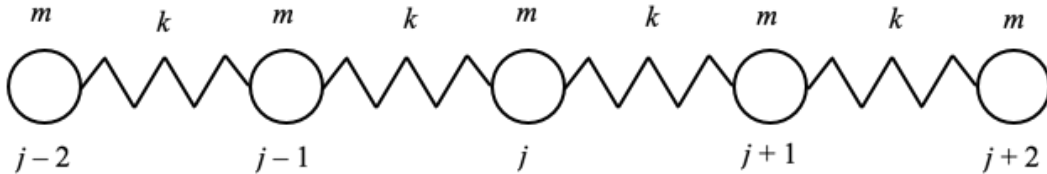


Figure 1: Portion of the infinite spring-mass chain.

Let u_j be the rightward displacement of the j^{th} mass, and a superposed dot denote time derivative. Recall that for linear springs, $f = k\delta$, where f is applied force and δ is spring extension. Then writing Newton's Second Law for the j^{th} mass gives:

$$m\ddot{u}_j = k(u_{j+1} - u_j) - k(u_j - u_{j-1}) \implies \ddot{u}_j - \frac{k}{m}(u_{j+1} - 2u_j + u_{j-1}) = 0. \quad (1)$$

We seek traveling wave solutions to Eq. (1) of the form

$$u_j(t) = Ue^{i(\omega t - qj)}, \quad (2)$$

where ω is wave frequency, $q = 2\pi/\lambda$ is wave number, λ is wavelength, and $c = \omega/q$ is phase speed - the speed of propagation of the wave, called phase speed to distinguish it from group speed (velocity). Substituting Eq. (2) into Eq. (1) gives:

$$-\omega^2 - \frac{k}{m}(e^{-iq} - 2 + e^{iq}) = 0. \quad (3)$$

Noting that

$$e^{-iq} - 2 + e^{iq} = 2(\cos q - 1) = -4 \sin^2 \frac{q}{2} \quad \left[\text{since } \cos q = \cos 2\left(\frac{q}{2}\right) = 1 - 2 \sin^2 \frac{q}{2} \right], \quad (4)$$

the solution to Eq. (3) is

$$\omega = 2\sqrt{\frac{k}{m}} \sin \frac{q}{2} \quad \implies \quad c = \frac{\omega}{q} = 2\sqrt{\frac{k}{m}} \frac{\sin \frac{q}{2}}{q}. \quad (5)$$

This shows that in the long wavelength limit, where $q \rightarrow 0$, that $c \rightarrow \sqrt{\frac{k}{m}}$.

Now, group velocity c_g is:

$$c_g = \frac{d\omega}{dq} = \sqrt{\frac{k}{m}} \cos \frac{q}{2} \quad \rightarrow \quad \sqrt{\frac{k}{m}} \quad \text{also as } q \rightarrow 0. \quad (6)$$

The usual way to write the dispersion relation for wave propagation is

$$\omega^2 = c^2 q^2. \quad (7)$$

When the phase speed c is constant (independent of wave number q , i.e. independent of wavelength λ), the wave is dispersionless. As just shown, this is the case in the long wavelength limit for the spring-mass system. However, as Eq. (5) shows, wave propagation for shorter wavelengths in the spring-mass system does involve dispersion, since then c indeed depends on q . This dependence is plotted in Figure 2. Note in the figure that c and c_g become equal and independent of q in the $\lambda \rightarrow \infty$ limit, but they otherwise decrease with decreasing wavelength. Note further that at $q = \pi$, meaning $\lambda = 2$ – that is, twice the mass spacing, $c_g = 0$, meaning that waves of shorter wavelength than $\lambda = 2$ – or higher frequency than $\omega = 2$ – will not propagate. Thus the system serves as a "low-pass" filter.

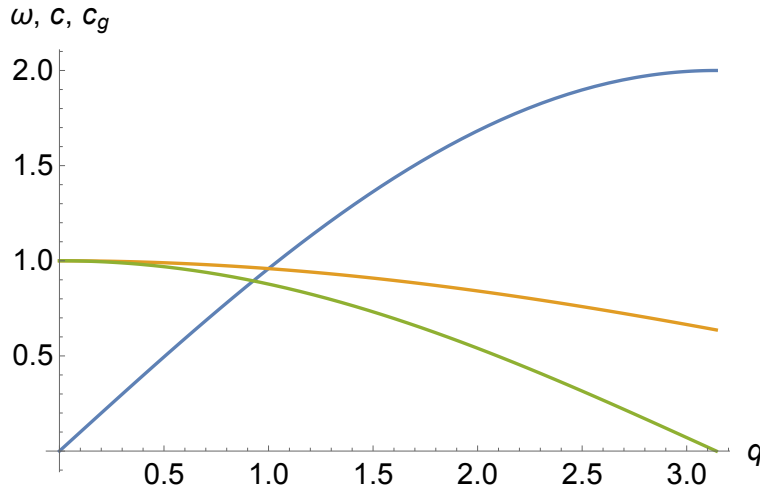


Figure 2: Plot of frequency (top), phase speed (center) and group speed (bottom) versus wave number, all normalized by $\sqrt{k/m}$.

2 Elastic Spring-Mass Chain with Damping

We next analyze an infinite spring-mass chain with viscous dampers. The chain is comprised of equal masses of mass m , equal springs of spring constant k , equal viscous dampers with viscous damping coefficient d , and unit spacing between the masses, as illustrated in Figure 3.

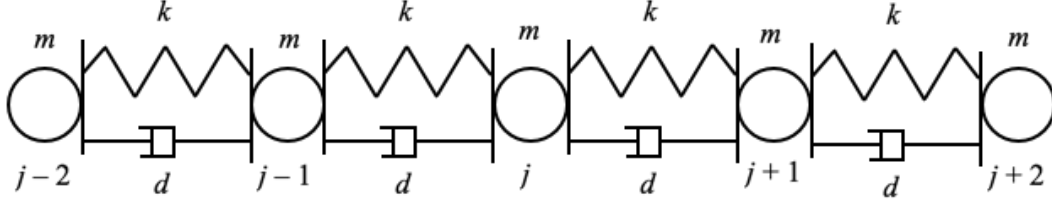


Figure 3: Portion of the infinite spring-mass-damper chain.

Recall that for linear viscous dampers, $f = d\dot{\delta}$. Writing Newton's Second Law for the j^{th} mass gives:

$$\begin{aligned} m\ddot{u}_j &= k(u_{j+1} - u_j) - k(u_j - u_{j-1}) + d(\dot{u}_{j+1} - \dot{u}_j) - d(\dot{u}_j - \dot{u}_{j-1}) \\ \implies \ddot{u}_j - \frac{k}{m}(u_{j+1} - 2u_j + u_{j-1}) - \frac{d}{m}(\dot{u}_{j+1} - 2\dot{u}_j + \dot{u}_{j-1}) &= 0. \end{aligned} \quad (8)$$

As before, we seek traveling wave solutions to Eq. (8). Now, however, we modify the form of Eq. (2) as follows, to allow for wave amplitude attenuation in time, where $\alpha = \omega + i\gamma$:

$$u_j(t) = Ue^{i(\alpha t - qj)}. \quad (9)$$

Substitution of Equation (9) into Equation (8) gives:

$$-\alpha^2 - \left(\frac{k}{m} + i\alpha\frac{d}{m}\right)(e^{-iq} - 2 + e^{iq}) = 0. \quad (10)$$

Again applying Equation (4), the solution to this is:

$$\alpha = 2 \sin \frac{q}{2} \left(\sqrt{\frac{k}{m} - \frac{d^2}{m^2} \sin^2 \frac{q}{2}} + i \frac{d}{m} \sin \frac{q}{2} \right), \quad (11)$$

so that we have found in the subcritical damping case, recalling that $\alpha = \omega + i\gamma$

$$\omega = 2 \sin \frac{q}{2} \sqrt{\frac{k}{m} - \frac{d^2}{m^2} \sin^2 \frac{q}{2}} \quad \text{and} \quad \gamma = 2 \frac{d}{m} \sin^2 \frac{q}{2}. \quad (12)$$

From these we have

$$c = 2 \frac{\sin \frac{q}{2}}{q} \sqrt{\frac{k}{m} - \frac{d^2}{m^2} \sin^2 \frac{q}{2}} \quad \text{and} \quad c_g = \cos \frac{q}{2} \sqrt{\frac{k}{m} - \frac{d^2}{m^2} \sin^2 \frac{q}{2}} \left[1 - \frac{\frac{d^2}{m^2} \sin^2 \frac{q}{2}}{\frac{k}{m} - \frac{d^2}{m^2} \sin^2 \frac{q}{2}} \right]. \quad (13)$$

To plot these as in Figure 2, we normalize both by $\sqrt{k/m}$, defining $d^* = \frac{d^2/m^2}{k/m} = \frac{d^2}{km}$, to obtain:

$$\frac{c}{\sqrt{k/m}} = 2 \frac{\sin \frac{q}{2}}{q} \sqrt{1 - d^* \sin^2 \frac{q}{2}} \quad \text{and} \quad \frac{c_g}{\sqrt{k/m}} = \cos \frac{q}{2} \sqrt{1 - d^* \sin^2 \frac{q}{2}} \left[1 - \frac{d^* \sin^2 \frac{q}{2}}{1 - d^* \sin^2 \frac{q}{2}} \right]. \quad (14)$$

These are plotted in Figure 4. Observe that beginning with $d^* \approx 0.5$, a reduction in the maximum wave number for positive group velocity occurs, with the largest reduction to about 1.55 for $d^* = 1$.

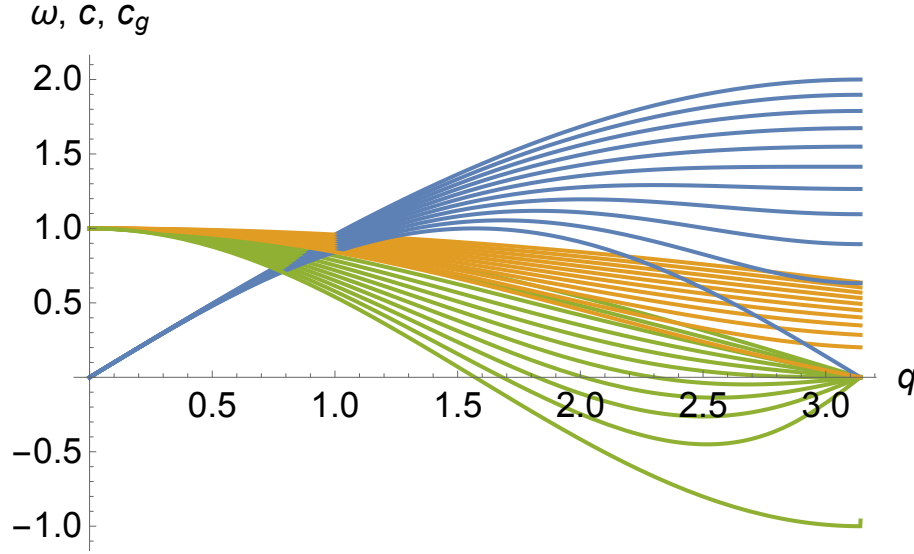


Figure 4: Plot of frequency, phase speed and group speed versus wave number, all normalized by $\sqrt{k/m}$, for $d^* = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]$. The uppermost curve in each is the zero damping ($d^* = 0$) case.

The damping ratio ζ is

$$\zeta = \frac{\text{Im}[\alpha]}{|\alpha|} = \frac{\gamma}{\sqrt{\omega^2 + \gamma^2}} = \frac{\frac{d}{m} \sin \frac{q}{2}}{\sqrt{k/m}} = \sqrt{d^*} \sin \frac{q}{2}; \quad (15)$$

this is plotted in Figure 5.

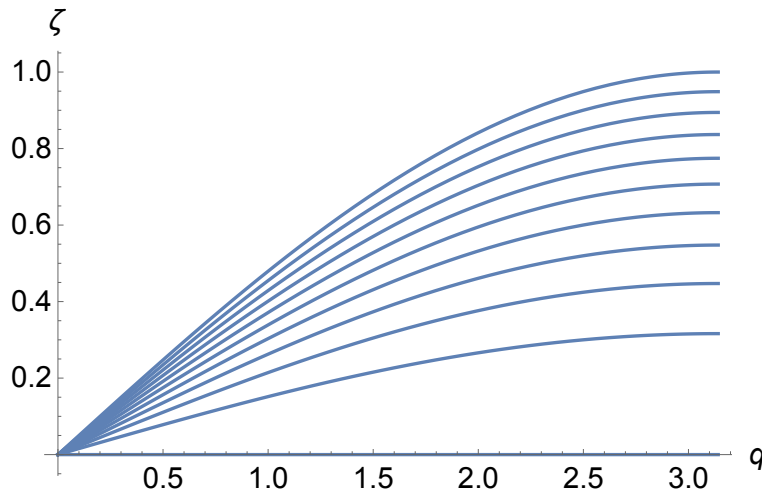


Figure 5: Plot of the damping ratio for $d^* = [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]$.

3 Spring-Mass-Damping Chain with Negative-Stiffness Elements

3.1 General Analysis

We now consider a chain of more complex elements, each of which involves three springs, three dampers and two masses arranged in such a way that one spring can have a negative stiffness range while the overall element remains stable. This chain is illustrated in Figure 6. For simplicity, we assume the spacing between unstarred masses to be unity, while that between starred and unstarred masses is $1/2$. By the j^{th} element, we will mean the mass labeled j and all components to the left

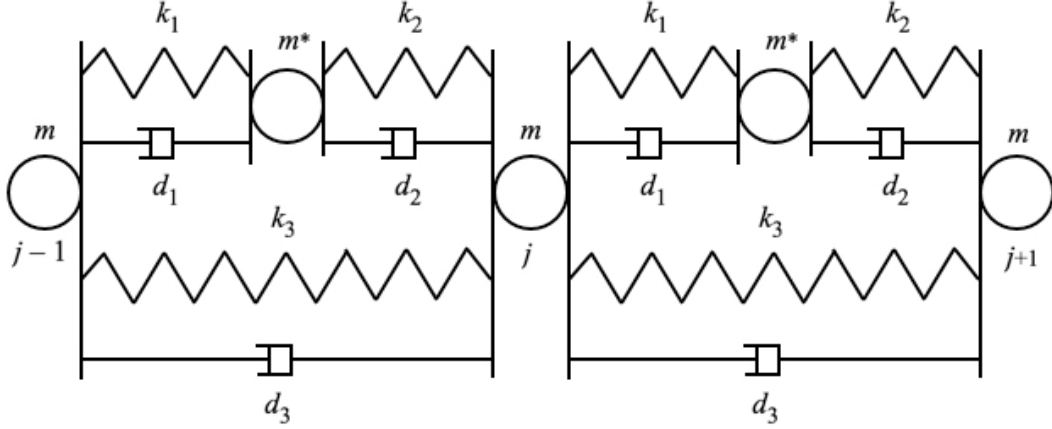


Figure 6: Portion of the infinite spring-mass-damper chain containing a negative-stiffness element with stiffness $k_1 < 0$.

of it up to but not including the $(j-1)^{st}$ mass; the rightward displacement of the m^* mass in the j^{th} element will be denoted by u_j^* . Writing Newton's Second Law for each of the two masses in the j^{th} element gives:

$$m^* \ddot{u}_j^* = k_2(u_j - u_j^*) - k_1(u_j^* - u_{j-1}) + d_2(\dot{u}_j - \dot{u}_j^*) - d_1(\dot{u}_j^* - \dot{u}_{j-1}) \quad (16a)$$

$$m \ddot{u}_j = k_1(u_{j+1}^* - u_j) - k_2(u_j - u_j^*) + d_1(\dot{u}_{j+1}^* - \dot{u}_j) - d_2(\dot{u}_j - \dot{u}_j^*) + k_3(u_{j+1} - 2u_j + u_{j-1}) + d_3(\dot{u}_{j+1} - 2\dot{u}_j + \dot{u}_{j-1}). \quad (16b)$$

As before, we seek traveling wave solutions to Eq. (16), of the form:

$$u_j(t) = U e^{i(\alpha t - q j)}, \quad u_j^*(t) = U^* e^{i[\alpha t - q(j-1/2)]}. \quad (17)$$

Substitution of (17) into (16) gives:

$$-\alpha^2 U^* e^{iq/2} - \left(\frac{k_2}{m^*} + i\alpha \frac{d_2}{m^*}\right)(U - U^* e^{iq/2}) + \left(\frac{k_1}{m^*} + i\alpha \frac{d_1}{m^*}\right)(U^* e^{iq/2} - U e^{iq}) = 0 \quad (18a)$$

$$-\alpha^2 U - \left(\frac{k_1}{m} + i\alpha \frac{d_1}{m}\right)(U^* e^{-iq/2} - U) + \left(\frac{k_2}{m} + i\alpha \frac{d_2}{m}\right)(U - U^* e^{iq/2}) - U \left(\frac{k_3}{m} + i\alpha \frac{d_3}{m}\right)(e^{-iq} - 2 + e^{iq}) = 0. \quad (18b)$$

Multiplying the first by $e^{-iq/2}$ and defining

$$K_n = k_n + i\alpha d_n, \quad (19)$$

equations (18) become

$$-\alpha^2 U^* - \frac{K_2}{m^*} (U e^{-iq/2} - U^*) + \frac{K_1}{m^*} (U^* - U e^{iq/2}) = 0 \quad (20a)$$

$$-\alpha^2 U - \frac{K_1}{m} (U^* e^{-iq/2} - U) + \frac{K_2}{m} (U - U^* e^{iq/2}) - U \frac{K_3}{m} (e^{-iq} - 2 + e^{iq}) = 0, \quad (20b)$$

or, gathering coefficients:

$$U \left(\frac{K_1}{m^*} e^{iq/2} + \frac{K_2}{m^*} e^{-iq/2} \right) + U^* \left(\alpha^2 - \frac{K_1}{m^*} - \frac{K_2}{m^*} \right) = 0 \quad (21a)$$

$$U \left[\alpha^2 - \frac{K_1}{m} - \frac{K_2}{m} + \frac{K_3}{m} (e^{-iq} - 2 + e^{iq}) \right] + U^* \left(\frac{K_1}{m} e^{-iq/2} + \frac{K_2}{m} e^{iq/2} \right) = 0. \quad (21b)$$

Nontrivial solutions of (16) of the form (17) require that the determinant of the coefficients of U and U^* in (21) vanish, which gives the following equation for α :

$$\begin{aligned} & \left(\alpha^2 - \frac{K_1}{m^*} - \frac{K_2}{m^*} \right) \left[\alpha^2 - \frac{K_1}{m} - \frac{K_2}{m} + \frac{K_3}{m} (e^{-iq} - 2 + e^{iq}) \right] \\ & - \left(\frac{K_1}{m^*} e^{iq/2} + \frac{K_2}{m^*} e^{-iq/2} \right) \left(\frac{K_1}{m} e^{-iq/2} + \frac{K_2}{m} e^{iq/2} \right) = 0. \end{aligned} \quad (22)$$

Substituting (19), (22) takes the following quartic form:

$$\alpha^4 + A\alpha^3 + B\alpha^2 + C\alpha + D = 0, \quad (23)$$

where the coefficients are

$$A = -\frac{i}{mm^*} \left[(m + m^*) (d_1 + d_2) + 4m^* d_3 \sin^2 \frac{q}{2} \right], \quad (24a)$$

$$B = -\frac{1}{mm^*} \left[(m + m^*) (k_1 + k_2) + 4 \sin^2 \frac{q}{2} (d_2 d_3 + d_1 d_2 + d_1 d_3 + m^* k_3) \right], \quad (24b)$$

$$C = \frac{i4}{mm^*} \sin^2 \frac{q}{2} [d_1 (k_2 + k_3) + d_2 (k_1 + k_3) + d_3 (k_1 + k_2)], \quad (24c)$$

$$D = \frac{4}{mm^*} \sin^2 \frac{q}{2} (k_2 k_3 + k_1 k_2 + k_1 k_3). \quad (24d)$$

3.2 System Stability Requirement on k_1

Since dampers do not affect the stability of the system, we may determine the restrictions on the allowable negative range of k_1 by setting $d_1 = d_2 = d_3 = 0$. The general quartic then reduces to a quadratic equation for α^2 , since $A = C = 0$:

$$\alpha^4 - \frac{(m + m^*) (k_1 + k_2) + 4 \sin^2 \frac{q}{2} m^* k_3}{mm^*} \alpha^2 + \frac{4 \sin^2 \frac{q}{2} (k_2 k_3 + k_1 k_2 + k_1 k_3)}{mm^*} = 0, \quad (25)$$

which has the solutions

$$\begin{aligned} \alpha^2 = & \frac{(m + m^*) (k_1 + k_2) + 4 \sin^2 \frac{q}{2} m^* k_3}{2mm^*} \\ & \pm \frac{\sqrt{[(m + m^*) (k_1 + k_2) + 4 \sin^2 \frac{q}{2} m^* k_3]^2 - 16mm^* \sin^2 \frac{q}{2} (k_2 k_3 + k_1 k_2 + k_1 k_3)}}{2mm^*}. \end{aligned} \quad (26)$$

Recalling our solution forms (17), stability requires that the mass displacements do not increase exponentially in time. This means that the imaginary part of α must be positive, which requires that the right side of (26) be positive. This reduces to the following simple requirement on the allowable negative range of k_1 for system stability:

$$\boxed{k_1 > -\frac{k_2 k_3}{k_2 + k_3}}. \quad (27)$$

3.3 System Behavior with No Damping

First we illustrate the behavior of this system with no damping, i.e., $d_1 = d_2 = d_3 = 0$. In this case, when the stability criterion (27) is satisfied, the positive square roots of (26) give the two system frequencies ω for each wave number q , so the dispersion curve $\omega(q)$ has two branches: the so-called acoustical and optical branches (having lower and higher frequencies, respectively). Here is what the dispersion curve and the group velocity looks like in the simple case $k_1 = k_2 = k_3 = k$, for three different values of the mass ratio:

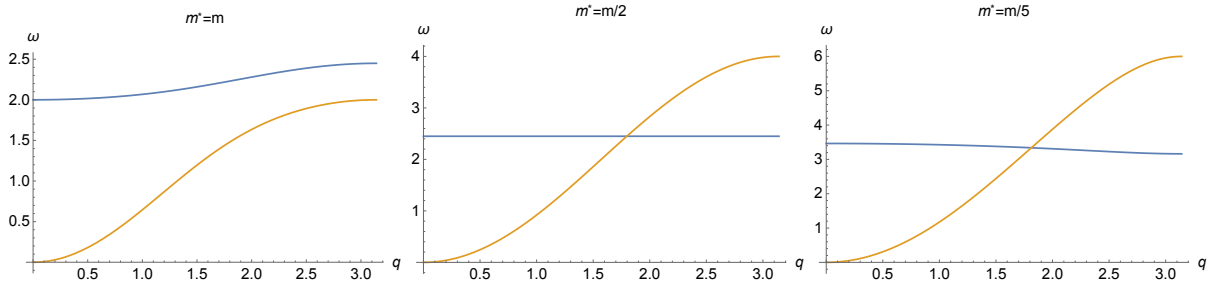


Figure 7: Plot of frequency normalized by $\sqrt{k/m}$, for $k_1 = k_2 = k_3 = k$, showing the effect of different mass ratios on dispersion curves.

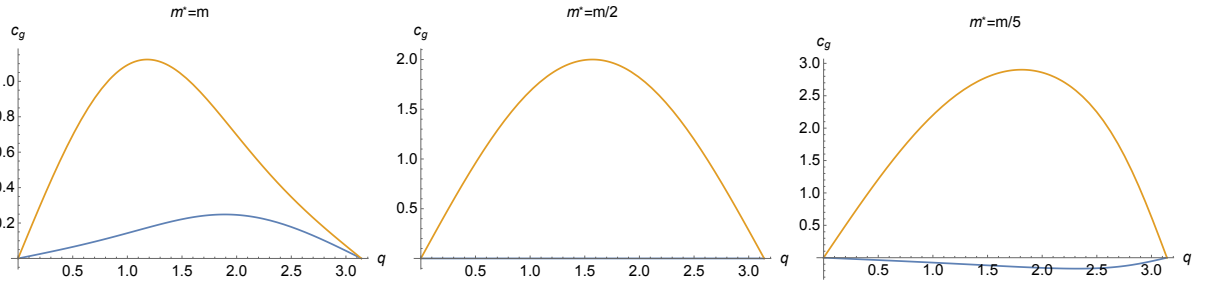


Figure 8: Plot of group velocity normalized by $\sqrt{k/m}$, for $k_1 = k_2 = k_3 = k$, showing the effect of different mass ratios.

Here are the effects of changing the spring constant ratios on the dispersion and group velocity curves.

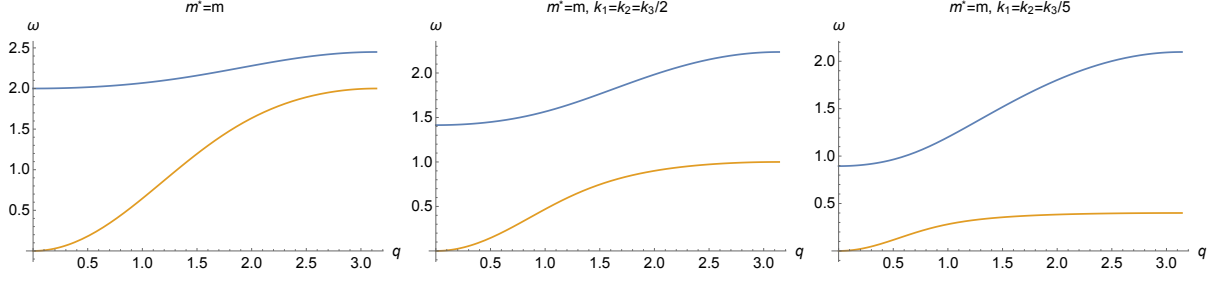


Figure 9: Effect of changing spring ratio on dispersion curves. Note that the no-pass band - the gap between the highest frequency of the lower curve and the lowest frequency of the higher curve - increases as the k -ratio increases.

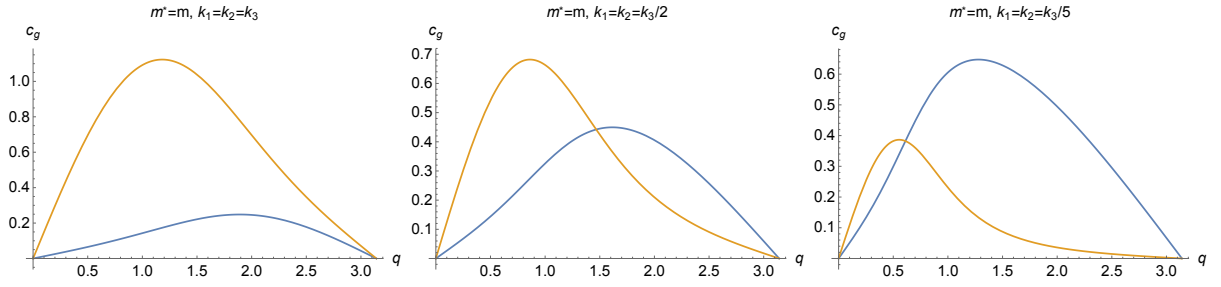


Figure 10: Effect of changing spring ratio on group velocity curves.

Here are the effects of changing both the mass and spring constant ratios simultaneously.

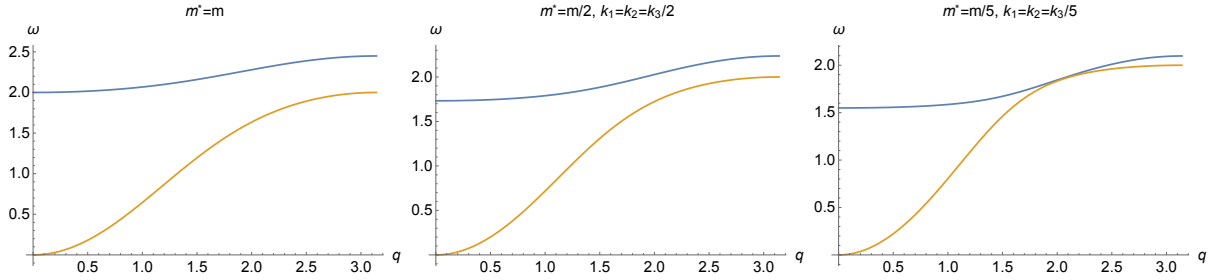


Figure 11: Effect of simultaneously changing mass and spring ratios on the dispersion curves.

Finally, we examine the effect of negative spring stiffness k_1 on both frequency and group velocity curves. Note that as the negative spring stiffness approaches the stability limit, the no-pass band is dramatically enlarged. Thus, even without consideration of damping, employing a negative-stiffness element provides a dramatic and potentially practically useful effect.

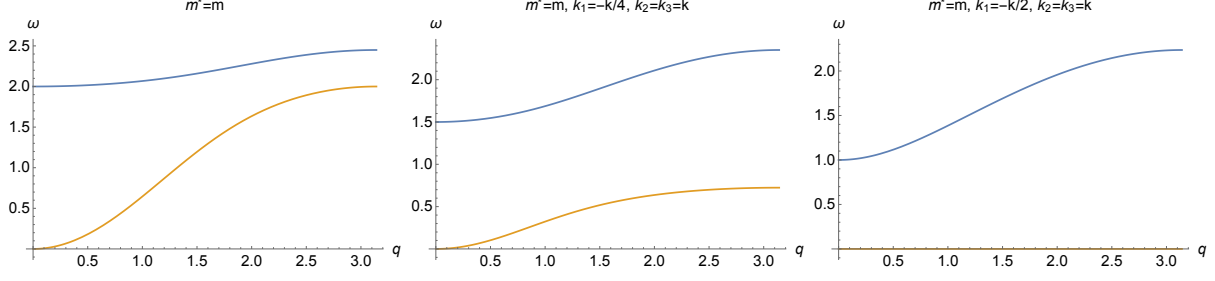


Figure 12: Effect of negative k_1 on dispersion curves; the right plot is at the stability limit. Notice how negative k_1 dramatically increases the no-pass band.

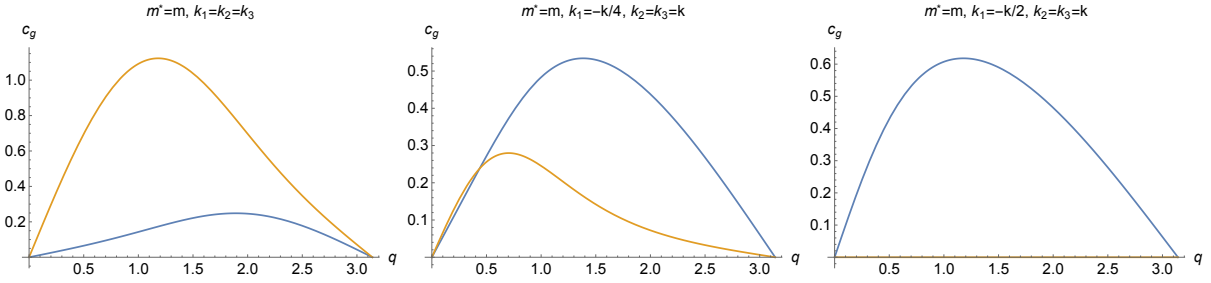


Figure 13: Effect of negative k_1 on group velocity curves.

3.4 System Behavior with Damping: Some Special Cases

3.4.1 $K_3 = 0$ Special Case

In the $K_3 = 0$ special case, $K_3 = 0, K_1 = K_2, m^* = m$, observe that (22) reduces to

$$\left(\alpha^2 - 2\frac{K_1}{m}\right)^2 = \frac{K_1^2}{m^2} 4 \cos^2 \frac{q}{2}, \quad (28)$$

taking the square root of which gives

$$\alpha^2 - 2\frac{k_1 + i\alpha d_1}{m} \left(1 \pm \cos \frac{q}{2}\right) = 0, \quad (29)$$

the solutions to which are

$$\alpha = 2 \sin \frac{q}{4} \left(i \frac{d_1}{m} \sin \frac{q}{4} \pm \sqrt{\frac{k_1}{m} - \frac{d_1^2}{m^2} \sin^2 \frac{q}{4}} \right), \quad \alpha = 2 \cos \frac{q}{4} \left(i \frac{d_1}{m} \cos \frac{q}{4} \pm \sqrt{\frac{k_1}{m} - \frac{d_1^2}{m^2} \cos^2 \frac{q}{4}} \right). \quad (30)$$

Observe that the first two solutions are identical to those we had found previously in (11) (modulo a change in spacing length from unity to $1/2$).

3.4.2 Long-Wavelength Limit

Observe that in the long-wavelength limit ($q \rightarrow 0$), (23) with (24) reduces to

$$\alpha^2 [mm^* \alpha^2 - i(m + m^*)(d_1 + d_2)\alpha - (m + m^*)(k_1 + k_2)] = 0, \quad (31)$$

which has the solutions

$$\alpha = 0, \quad \alpha = i \frac{m + m^*}{2mm^*} (d_1 + d_2) \pm \frac{\sqrt{m + m^*}}{2mm^*} \sqrt{4mm^*(k_1 + k_2) - (m + m^*)(d_1 + d_2)^2}. \quad (32)$$

These show that the only nonzero long-wavelength frequency is

$$\omega = \frac{\sqrt{m + m^*}}{2mm^*} \sqrt{4mm^*(k_1 + k_2) - (m + m^*)(d_1 + d_2)^2}. \quad (33)$$

This shows that for $k_1 = k_2 = k, d_1 = d_2 = d, m^* = m, \omega \rightarrow 0$ when $d^* \rightarrow 1$.

More generally, for smaller d^* , (33) shows that ω can be made zero by choosing m^* such that [assuming of course $4m(k_1 + k_2) - (d_1 + d_2)^2 > 0$]

$$m^* = \frac{m(d_1 + d_2)^2}{4m(k_1 + k_2) - (d_1 + d_2)^2}. \quad (34)$$

Alternatively, permitting a negative-stiffness spring means that k_1 can be chosen as follows to make $\omega = 0$ in the long-wavelength limit:

$$k_1 = -k_2 + \frac{(m + m^*)(d_1 + d_2)^2}{4mm^*}, \quad (35)$$

making sure that the stability requirement (27) is met. Thus, these are all ways to adjust the system so that $\omega = 0$ in the long-wavelength limit.

Eq. (32) also shows that the long-wavelength damping ratio ζ is

$$\zeta = \frac{\text{Im}[\alpha]}{|\alpha|} = \frac{1}{2} \sqrt{\frac{m + m^*}{mm^*}} \frac{d_1 + d_2}{\sqrt{k_1 + k_2}}. \quad (36)$$

This shows very clearly that for any given sets of masses and damping coefficients, permitting stable negative k_1 can substantially increase the damping of long-wavelength waves. Specifically, (27) with (36) show that the strategy that yields the highest damping of long-wavelength waves is to choose k_2 to be as small as possible, and very small compared to k_3 , and then to choose k_1 to be as close to $-k_2$ as (27) permits. For example, if $k_2 = k_3/10$, then we may choose k_1 just slightly larger than $-\frac{10}{11}k_2$ without violating stability, which will increase ζ by over a factor of 3 compared to the case $k_1 = 0$.

3.5 General Solution to Full System

We now return to the general analysis of Subsection 3.1. The general quartic equation for α , (23), has analytical expressions for its four roots. Defining

$$p = 2B - \frac{3}{4}A^2, \quad r = \frac{A^3}{4} - AB + 2C, \quad (37a)$$

$$\Delta_0 = B^2 - 3AC + 12D, \quad \Delta_1 = 2B^3 - 9(ABC - 3A^2D - 3C^2 + 8BD), \quad (37b)$$

$$Q = 2^{-\frac{1}{3}} \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}, \quad S = \frac{1}{\sqrt{3}} \sqrt{Q + \frac{\Delta_0}{Q} - p}, \quad (37c)$$

the roots are:

$$\alpha_{1,2} = \frac{1}{2} \left(S - \frac{A}{2} \pm \sqrt{-\frac{r}{S} - S^2 - p} \right), \quad \alpha_{3,4} = \frac{1}{2} \left(-S - \frac{A}{2} \pm \sqrt{\frac{r}{S} - S^2 - p} \right). \quad (38)$$

The real parts of the first two roots, $\alpha_{1,2}$, give the positive system frequencies. [The real parts of $\alpha_{3,4}$ are the negatives of these.] These are identical to the positive square roots of (26) when there is no damping.

Recall that in general $\alpha = \omega + i\gamma$. Thus the frequency $\omega(q)$ and damping ratio $\zeta(q)$ of the acoustical and optical branches can be obtained from $(38)_1$ as:

$$\omega(q) = \text{Re}[\alpha(q)], \quad \zeta(q) = \frac{\text{Im}[\alpha(q)]}{|\alpha(q)|}. \quad (39)$$

3.6 Work in Progress

We are currently employing the general solution to the full system just summarized to explore and optimize the effects of stable negative stiffness on wave propagation in the full lumped system.

References

- [1] Brillouin, L., 1953, *Wave Propagation in Periodic Structures*, Dover, N.Y.
- [2] Born, M. and Huang, K., 1954, *Dynamical Theory of Crystal Lattices*, Clarendon Press, Oxford.